ON THE LOADING FUNCTIONS OF ANISOTROPICALLY HARDENING PLASTIC MATERIALS

(O FUNKTSIIAKH NAGRUZHENIIA ANIZOTROPNO UPROCHNIAIUSHCHEGOSIA PLASTICHESKOGO MATERIALA)

PMM Vol.28, № 4, 1964, pp. 794-797

G.I.BYKOVTSEV, V.V.DUDUKALENKO and D.D.IVLEV
(Voronezh)

(Received January 28, 1964)

1. Let us assume, that the loading function of a hardening plastic material is determined entirely by the state of stress and strain of the material

$$f(\sigma_{ij}, e_{ij}^p) = 0 \tag{1.1}$$

where σ_{ij} are the components of the state of stress, e_{ij}^p are the components of plastic strain.

Let us assume, that the given path of loading leads to a determined deformed state independently of the orientation of the body with respect to some Cartesian system of coordinates x, y, z. Then the loading function (1.1) can depend only on the invariants of the state of stress and strain. The invariants of the state of stress and strain will be the invariants of the tensors σ_{ij}, e_{ij}^p and also the common invariants of these tensors.

It is known (for example, [1]), that the number of basic invariants, through which all the invariants of the tensors σ_{ij} and e_{ij}^p can be expressed (including also the common ones) is equal to nine. This circumstance corresponds to the fact, that the given state of stress and strain is determined entirely by six values of the principal components of the state of stress and strain and also by three independent values, which characterize a common orientation of the principal directions of the tensors σ_{ij} and e_{ij}^p.

In this way, one can write

$$f(\sigma_i, e_i^p, \alpha, \beta, \gamma) = 0 \qquad (i = 1, 2, 3) \tag{1.2}$$

where σ_i and e_i^p are the principal components of the stress tensor and of the plastic strain tensor; α, β and γ are three values, for example the Eulerian angles, characterizing a common orientation of the principal directions of σ_i and e_i^p.

Any nine independent invariants can be chosen as basic, for example

$$\begin{gathered}\delta_{ij}\sigma_{ij}, \quad \sigma_{ij}\sigma_{ji}, \quad \sigma_{ij}\sigma_{jk}\sigma_{ki}, \quad \delta_{ij}e_{ij}^p, \quad e_{ij}^p e_{ji}^p \\ e_{ij}^p e_{jk}^p e_{ki}^p, \quad \sigma_{ij}e_{ij}^p, \quad \sigma_{ij}e_{jk}^p e_{ki}^p, \quad \sigma_{ij}\sigma_{jk}e_{ki}^p\end{gathered} \tag{1.3}$$

where δ_{ij} are the components of the unit tensor.

2. Let us consider some singularities of the behavior of the loading function (1.2). For simplicity let us restrict ourselves to the shear case. Let us assume, that the only components different from zero are τ_{xz}, τ_{yz}, e_{xz}^p, e_{yz}^p. Further on we will omit z and p indices.

The state of stress and strain in shear can be represented by the vectors

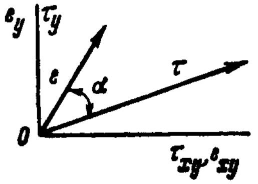

Fig. 1

$$\tau = \tau_x \mathbf{i} + \tau_y \mathbf{j}, \qquad \mathbf{e} = e_x \mathbf{i} + e_y \mathbf{j} \tag{2.1}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors. The invariants of the state of stress and strain are

$$\tau^2 = \tau_x^2 + \tau_y^2, \qquad e^2 = e_x^2 + e_y^2 \tag{2.2}$$

$$\alpha = \tan^{-1} \frac{\tau_x e_y - \tau_y e_x}{\tau_x e_x + \tau_y e_y} \tag{2.3}$$

where α is the angle between the vectors τ and $\mathbf{e}$, (Fig.1).

Instead of the angle α it is convenient to use one of the invariants

$$\tau_x e_x + \tau_y e_y, \qquad |\tau_x e_y - \tau_y e_x| \tag{2.4}$$

It is evident, that any three of the invariants (2.2), (2.4) can be used as basic. Let us examine the loading function

$$f(\tau_x^2 + \tau_y^2, e_x^2 + e_y^2, |\tau_x e_y - \tau_y e_x|) = 0 \tag{2.5}$$

According to (2.5) any initial yielding curve in the plane τ_x, τ_y will be a circle. Indeed, $e_x = e_y = 0$ at the initial yielding moment condition (2.5) has the form

$$f(\tau_x^2 + \tau_y^2, 0, 0) = 0 \tag{2.6}$$

Let us assume that the loading $\tau_x \neq 0$, $\tau_y = 0$ takes place. If the loading function (2.5) remains symmetrical with respect to the τ_x-axis, then $e_x \neq 0$, $e_y = 0$. The dependence between τ_x and e_x will be determined from condition (2.5)

$$f(\tau_x^2, e_x^2, 0) = 0 \tag{2.7}$$

The loading function (2.5), which corresponds to the condition $\tau_x \neq 0$, $e_x \neq 0$, $\tau_y = e_y = 0$ has the form

$$f(\tau_x^2 + \tau_y^2, e_x^2, |\tau_y e_x|) = 0 \tag{2.8}$$

Let us assume, that the loading function (2.5) does not depend on the invariant $e_x^2 + e_y^2$,

$$f(\tau_x^2 + \tau_y^2, |\tau_x e_y - \tau_y e_x|) = 0 \tag{2.9}$$

Then during the loading $\tau_x \neq 0$, $e_x \neq 0$, $\tau_y = e_y = 0$, Expression (2.9) will take the form $f(\tau_x^2, 0) = 0$, and, therefore, during the initial loading the material does not harden in the direction of loading.

However the loading function will change during the change of the state of strain. Indeed, the loading function (2.9), which corresponds to the state of strain $e_x \neq 0$, $e_y = 0$, will have the form

$$f(\tau_x^2 + \tau_y^2, |\tau_y e_x|) = 0 \tag{2.10}$$

As an example, let us examine the loading function

$$\tau_x^2 + \tau_y^2 = k^2 \pm c^2 (\tau_x e_y - \tau_y e_x)^2 \qquad (k, c = \text{const} > 0) \tag{2.11}$$

Let us consider again a uniaxial loading $\tau_x \neq 0$, $e_x \neq 0$, $\tau_y = e_y = 0$. From (2.11) it follows, that $\tau_x = k$, therefore, the material does not harden along the direction of loading, and the loading function has the form

$$\tau_x^2 + \tau_y^2 (1 \mp c^2 e_x^2) = k^2 \tag{2.12}$$

Equation (2.12) in the plane τ_x, τ_y will be an equation of an ellipse with the semiaxes k and $k / \sqrt{1 \mp (ce_x)^2}$. The ellipses are shown in Fig. 2, representing the loading curves (2.12), the initial yielding curve is represented by dashed line. The ellipse with the increasing semiaxis corresponds to the upper sign in Equations (2.11) and (2.12), and the ellipse with the decreasing semiaxis corresponds to the lower sign in the same equations.

In this way, the dependence of the loading function on the invariant $\tau_x e_y - \tau_y e_x$, can lead to interesting consequences: a material nonhardening in the direction of the

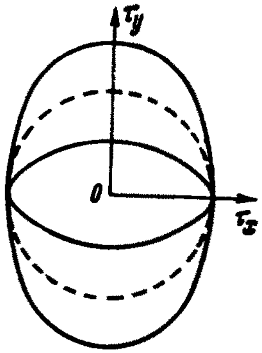

Fig. 2

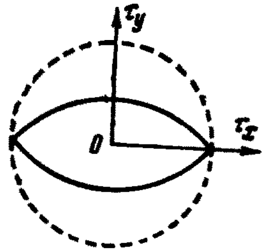

Fig. 3

initial loading can harden or unharden in other directions. An interesting case is represented by the relation

$$\tau_x^2 + \tau_y^2 = k^2 + 2c\,|\,\tau_x e_y - \tau_y e_x\,| \tag{2.13}$$

In this case let us consider also the loading $\tau_x \neq 0,\ \tau_y = 0$, under which $e_x \neq 0,\ e_y = 0$, takes place. The loading function will have the form

$$\tau_x^2 + (\tau_y \mp c e_x)^2 = k^2 + (c e_x)^2 \tag{2.14}$$

The loading curve which corresponds to the loading function (2.14), is represented in Fig.3. In this case the material does not harden in the direction of loading, and the loading function acquires an angular point.

Let us assume, that the curve $f(x, y, a) = 0$ is given in the plane x, y, where a is a parameter.

It is easy to construct a loading function, represented by a loading curve which for $\tau_x \neq 0,\ e_x \neq 0,\ \tau_y = e_y = 0$ will coincide with the curve $f(x, y, a) = 0$ given in advance; such that for $a = 0$ the curve $f(x, y, 0) = 0$ is a circle.

Indeed, let us make a correspondence between τ_x, τ_y, e_x and x, y, a respectively. The sought loading function can be chosen, for example, in the form

$$f\left(\frac{\tau_x e_x + \tau_y e_y}{\sqrt{e_x^2 + e_y^2}},\ \frac{|\tau_x e_y - \tau_y e_x|}{\sqrt{e_x^2 + e_y^2}},\ \sqrt{e_x^2 + e_y^2}\right) = 0 \tag{2.15}$$

or

$$f\left(\sqrt{\tau_x^2 + \tau_y^2},\ \frac{|\tau_x e_y - \tau_y e_x|}{\sqrt{e_x^2 + e_y^2}},\ \sqrt{e_x^2 + e_y^2}\right) = 0 \tag{2.16}$$

Both loading functions (2.15) and (2.16) lead to the same loading curve when the load is $\tau_x \neq 0,\ e_x \neq 0,\ \tau_y = e_y = 0$, but for the repeated loadings in other directions their behavior will be different.

In the same way one can find other loading functions, which for uniaxial loading lead to the given loading curve.

Let us make several observations:

a) The consideration of repeated loadings in other directions is connected with a compulsory use of the associated law of plastic flow. Depending on the choice of the loading function (2.5) different effects can be described: rotation of the loading curve, Bauschinger's transversal effect, etc.

b) In the same way, the plane problem can be considered for a noncompressible plastic material. In this case the state of stress and strain condition can be represented by the vectors

$$\sigma = {}^1/_2\,(\sigma_x - \sigma_y)\,\mathbf{i} + \tau_{xy}\mathbf{j}, \qquad \mathbf{e} = {}^1/_2\,(e_x - e_y)\,\mathbf{i} + e_{xy}\mathbf{j} \tag{2.17}$$

The consideration of the state of stress and strain condition is related with the known difficulties of interpretation.

c) It is obvious, that the theory of translational hardening [2 and 3] represents a particular case of the above considered relations; in this case the loading function will have the form

$$(\tau_x - ce_x)^2 + (\tau_y - ce_y)^2 = k^2 \tag{2.18}$$

or

$$(\tau_x{}^2 + \tau_y{}^2) + (e_x{}^2 + e_y{}^2) - 2c\,(\tau_x e_x + \tau_y e_y) = k^2 \tag{2.19}$$

BIBLIOGRAPHY

1. Gol'denblatt, I.I., Nekotorye voprosy mekhaniki deformiruemykh sred (Some Problems of Mechanics of Deformable Media). Gostekhteoretizdat, 1955.
2. Ishlinskii, A.Iu., Obshchaia teoriia plastichnosti s lineinym uprochneniem (General theory of plasticity with linear hardening). Ukr.matemat. zh., Vol.6, № 3, 1954.
3. Kadashevich, Iu.I. and Novozhilov, V.V., Teoriia plastichnosti, uchityvaiushchaia ostatochnye mikronapriazheniia (Theory of plasticity which considers residual microstresses). *PMM* Vol.22, № 1, 1958.

Translated by E.Z.